ELSEVIER
Discrete Applied Mathematics 72 (1997) 71–84
DISCRETE APPLIED MATHEMATICS

# Comparability graph augmentation for some multiprocessor scheduling problems

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Received 21 December 1993; revised 12 October 1994

**Abstract**

A comparability graph is a graph which admits a transitive orientation. In this paper we consider the problem of augmenting a graph to a comparability graph in such a way that the maximum weight of its cliques is minimum. The problem is equivalent to a multiprocessor scheduling problem and to the interval coloring problem; and in the unweighted case also to the chromatic number problem. In the general case, the problem is NP-hard in the strong sense even on some very simple types of perfect graphs. We give complexity and approximation results for two subclasses of perfect graphs, namely for split graphs and stars of cliques, for which the problem still remains intractable but admits efficient estimations.

*Keywords*: Multiprocessor scheduling; Interval coloring; Comparability graphs; Split graphs; Computational complexity; Approximation results

## 1. Introduction

We investigate the problem of finding an augmentation of a vertex-weighted graph to a comparability graph in such a way that the maximum weight of cliques is minimized. We will refer to this problem as the Comparability Augmentation Problem. In the unweighted case the problem is equivalent to the chromatic number problem and thus can be solved in polynomial time on perfect graphs e.g. by the ellipsoid method (see [8]), and also by elementary combinatorial algorithms on several classes of perfect graphs (cf. e.g. [7]). On the other hand, in the weighted case, the problem is NP-hard in the strong sense even on some rather restricted classes of graphs (see [11, 2]).

In this paper we prove NP-hardness results for some further simple particular classes of perfect graphs, namely for split graphs and stars of cliques. Moreover, for these

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PII S0166-218X(96)000037-6

classes of graphs we obtain tight upper bounds on the optimum value of the objective function, telling how much the weight of the maximum weighted clique, which is easy to compute, can increase when these graphs are augmented to comparability graphs. In particular, it is shown that on split graphs the clique can increase at most by a multiplicative constant 2, while on stars of cliques it can increase at most by 1.5.

### 1.1. Motivations and background

The interest in the Comparability Augmentation Problem comes from its equivalence with a scheduling problem and the Interval Coloring Problem.

The scheduling problem, which will be referred to as $P|\text{fix}|C_{\max}$ (according to the notation introduced in Hoogeveen, [9]), is defined as follows. Given a set of dedicated resources, each of which can be used by a single task at a time, and a set of tasks, each of which requires a subset of the resources for a given processing time, the problem is to determine a starting time for each task so that the maximum completion time of the tasks (makespan) is minimized. A *normal schedule* is a solution of the scheduling problem in which all tasks requiring the same set of resources are scheduled consecutively. This problem has been dealt in a pioneering paper by Bozoki and Richard [3] and, more recently, by Kubale [11], Bianco et al. [1], Čangalović and Schreuder [4], Blazewicz et al. [2], Dell'Olmo et al. [5], Hoogeveen et al. [9]. A problem instance can be represented by a graph, called constraint graph, in which the vertices are associated with the tasks and an edge is drawn between two tasks whenever the tasks share some resource. Moreover, the processing time of the task is associated to each vertex. The problem of minimizing the makespan in the scheduling problem becomes the problem of finding an acyclic orientation of the constraint graph so that the path of maximum length is minimized. One can see that the transitive closure of the oriented graph is a comparability graph in which the maximum weighted clique has the same weight as the heaviest path. Thus, the scheduling problem is equivalent to the Comparability Augmentation Problem.

The following scheduling problem motivates, in the scheduling context, the investigation of the clique stars. We are given a set of $n$ independent tasks $\{T_1,\dots,T_n\}$, with processing times $\{p_1,\dots,p_n\}$, and a set of $m+1$ dedicated machines $\{M_0,M_1,\dots,M_m\}$. Each task must be processed either on a single machine $M_i$, or on a pair of machines $(M_0,M_i)$, with $i=1,\dots,m$. The graph associated to this problem is a clique star, later formally defined. A "central" clique $C_0$ corresponds to the tasks requiring two machines and each other maximal clique $C_i$ corresponds to the tasks using machine $M_i$, either alone or together with machine $M_0$.

A different scheduling problem gives rise to a split graph. Consider the scheduling problem in which, in addition to regular work, a special operation (e.g. maintenance) must be carried out (by a person or a group of persons) on a certain number of independent resources (e.g. airplanes, machines) before or after each resource has been used for the regular work. This problem is represented by means of a split graph, in which the vertices of the clique represent the special operations on the resources and each vertex of the independent set represents the regular work on a resource. A more general split graph is obtained in the case the special operation can be carried out simultaneously on more than one resource and/or an additional resource is needed for a special operation and the regular work on a different resource.

A different graph-theoretical formulation of the scheduling problem is obtained as follows. The $P|\text{fix}|C_{\max}$ problem is the problem of assigning to each vertex an interval of colors — that is, a set of consecutive colors — whose length is equal to the processing time so that the total number of colors is minimized. This is exactly the Interval Coloring Problem (see, for instance, [7]).

Moreover, if the constraint graph has a particular structure, the $P|\text{fix}|C_{\max}$ problem is equivalent to other well-known problems. In particular, if the graph is an interval graph, the problem is equivalent to the Dynamic Storage Allocation Problem (see [6]), which is defined as follows. Given a set of rectangles of different lengths and heights which can be moved vertically but not horizontally, the problem is to find their position in such a way that the vertical difference between the highest and lowest points of the rectangles is as small as possible. First of all, note that this problem is equivalent to the $P|\text{fix}|C_{\max}$ problem in the case where a linear order of the resources exists such that the resources required by each task are consecutive. Then, observe that the constraint graph of such an instance has the following property. Consider the maximal cliques of the graph, each of which contains all the tasks requiring a particular resource, and order them in the same linear order used for the resources. (If such a clique is not maximal, it can be replaced by the maximal clique which is the union of a consecutive set of the cliques.) It follows that, for each task, as it requires a consecutive number of resources, the maximal cliques containing the task occur consecutively. This is one of the characterizations of interval graphs (see [7]). Conversely, take an interval graph and order the cliques in such a way that, for each vertex, the maximal cliques containing the vertex occur consecutively. Then, associate to each maximal clique a resource so that a task requires all the resources associated to the cliques it belongs to. It follows that each task of a so built instance of the $P|\text{fix}|C_{\max}$ problem requires a consecutive set of resources. Then the equivalence between the Dynamic Storage Allocation Problem and the $P|\text{fix}|C_{\max}$ problem on interval graphs is shown.

As we have already mentioned, in the weighted case the Comparability Augmentation Problem is NP-hard in the strong sense. The only known case in which both the weighted (interval) and the unweighted (usual) coloring problems can be solved in polynomial time is on comparability graphs. In this case the Comparability Augmentation Problem reduces to the problem of finding the maximum weighted clique of the graph, which can be obtained in polynomial time on comparability graphs.

## 2. Notation and terminology

A *graph* is a pair $\mathscr{G} = (V, E)$, where $V$ is a finite set of $n = |V|$ elements called *vertices* and $E \subseteq \{(x, y): x, y \in V, x \neq y\}$ is a set of $e = |E|$ *unordered* vertex pairs called *edges*. For distinct vertices $x, y$ we say that $x$ is *adjacent* to $y$ (or equivalently, $y$ is adjacent to $x$) if $(x, y) \in E$. Otherwise, they are said to be *independent*. A graph is a *triangulated graph*, or *chordal graph*, if it contains no induced cycle of length greater than 3 (i.e., no subgraph with vertex set $\{x_1, \ldots, x_k\}$, $k \geqslant 4$, and edges $(x_i, x_j)$ if and only if $j = i + 1$ or $(i, j) = (1, k)$).

If we orient an existing edge $(x, y)$ of $\mathscr{G}$ from $x$ to $y$, we write $x \to y$. A *transitive orientation* of a graph is obtained by orienting all its edges so that if $x \to y$ and $y \to z$, then $x \to z$. If a graph admits a transitive orientation, it is called a *comparability graph*. Comparability graphs can be recognized and transitively oriented in polynomial time, and have a number of other interesting properties; see [7, 10] for a survey.

A set $V' \subseteq V$ of vertices is a *clique* if the vertices in $V'$ are pairwise adjacent. A *maximum clique* is one with largest number of vertices among all cliques. A *path* of $\mathscr{G}$ is a sequence of vertices $v_1, \ldots, v_p$ such that $v_i$ is adjacent to $v_{i+1}$, for $i = 1, \ldots, p-1$. A set $V' \subseteq V$ is an *independent set* if the vertices of $V'$ are pairwise independent. A *maximum independent set* is one with largest number of vertices among all independent sets. A *split graph* is a graph whose set $V$ of vertices is partitioned into two subsets, $V = K \cup S$, such that $K$ is a clique and $S$ is an independent set.

A family $\{C_1, \ldots, C_k\}$ of sets of vertices is called a *coloring* of $\mathscr{G}$ if $C_i \cap C_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{k} C_i = V$, and $C_i$ is an independent set for each $i$. A minimum coloring is one with the smallest number of sets $C_i$ among all colorings. The number of sets $C_i$ in the minimum colorings is called the *chromatic number* of $\mathscr{G}$.

A *weighted graph* $\mathscr{G}$ is a graph with an associated weight function $w : V \to N \cup \{0\}$ which assigns a nonnegative integer weight to each vertex of $V$. For vertex $x$, $w(x)$ is the weight of $x$. (Vertices of zero weight are irrelevant in the context of scheduling and Comparability Augmentation, but technically it is convenient to allow their presence in some cases.) The weight $w(S)$ of a set $S \subset V$ is defined as $\sum_{x \in S} w(x)$. A maximum weighted clique of $\mathscr{G}$ is a clique with largest weight among all the cliques. For an integer $t \geqslant 2$, assuming that all weights are positive, an *interval t-coloring*, or shortly *interval coloring*, of $\mathscr{G}$ is a function $c : V \to \{0, 1, \ldots, t-1\}$ such that $c(x) + w(x) \leqslant t$ and, if $c(x) < c(y)$ and $x$ is adjacent to $y$, then $c(x) + w(x) \leqslant c(y)$. An interval coloring $c$ of $\mathscr{G}$ can be viewed as the assignment of an "interval" $\{c(x), c(x)+1, \ldots, c(x)+w(x)-1\}$ of $w(x)$ colors to each vertex $x$ so that the intervals of colors assigned to two adjacent vertices do not overlap. The *interval chromatic number* of $\mathscr{G}$ is the minimum $t$ such that $\mathscr{G}$ has an interval coloring.

We say that $\mathscr{G}^+ = (V, E \cup F)$ is an *augmentation* of $\mathscr{G} = (V, E)$ if $\mathscr{G}$ and $\mathscr{G}^+$ have the same set $V$ of vertices and $\mathscr{G}^+$ has a larger set of edges than $\mathscr{G}$. An augmentation of $\mathscr{G}$ to a comparability graph is one such that the augmented graph $\mathscr{G}^+$ admits a transitive orientation.

The *Comparability Augmentation Problem* is defined as follows. Given a weighted graph $\mathscr{G} = (V, E)$, the problem is to find an augmentation of $\mathscr{G}$ to a comparability graph $\mathscr{G}^+$ such that the maximum weight of the cliques in $\mathscr{G}^+$ is as small as possible. This minimum value of the maximum weights is denoted by $M^*(\mathscr{G})$. Moreover, we denote by $C_{\max}(\mathscr{G})$ the largest weight of a clique in $\mathscr{G}$. Clearly, $M^*(\mathscr{G}) \geqslant C_{\max}(\mathscr{G})$ holds for every graph $\mathscr{G}$.

We now define some particular classes of graphs whose weighted versions, called for the sake of simplicity with the same name, will be investigated in the paper.

A *clique star* is a connected graph in which the maximal cliques, denoted by $C_0, C_1, \ldots, C_m$, satisfy the following properties: $C_0 \cap C_i \neq \emptyset$ for all $1 \leqslant i \leqslant m$, $C_i \cap C_j = \emptyset$ for all $1 \leqslant i < j \leqslant m$, and $C_0 \subset C_1 \cup \cdots \cup C_m$. It is obvious that every clique star is a triangulated graph. We denote by $\mathscr{W}$ the class of those clique stars in which $m = 3$.

Finally, let us define a small subclass of split graphs which also are particular clique stars. We denote by $S^n$ the graph with vertex set $V = \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$ and edge set $\{(a_i, b_i)\colon\ 1 \leqslant i \leqslant n\} \cup \{(a_i, a_j)\colon\ 1 \leqslant i < j \leqslant n\}$. Some examples are shown in Figs. 1(a), 1(b) and 3(a).

## 3. NP-completeness results

The proofs of our two NP-completeness theorems will be based on a reduction from the Partition Problem, defined as follows:

*Instance:* A set of natural numbers $w_1, \ldots, w_n$ with $\sum_{i=1}^{n} w_i = 2s$.

*Question:* Does there exist a partition of $\{w_i\colon\ 1 \leqslant i \leqslant n\}$ into two subsets $A_1$ and $A_2$ such that $\sum_{w_i \in A_1} w_i = \sum_{w_j \in A_2} w_j = s$ ?

The graphs $W \in \mathscr{W}$ have the following structure: the vertex set is partitioned in six subsets $A^1$, $A^2$, $A^3$, $B^1$, $B^2$ and $B^3$, say $a_i^j$ and $b_i^j$ are the vertices of the sets $A^j = C_j \setminus C_0$ and $B^j = C_j \cap C_0$, $j = 1, 2, 3$, respectively; and the edge set is $\{(b_i^j, b_{i'}^{j'})\ :\ (i,j) \neq (i',j'); 1 \leqslant j, j' \leqslant 3\} \cup \{(a_i^j, a_{i'}^j)\colon\ i \neq i'; j = 1, 2, 3\} \cup \{(b_i^j, a_{i'}^j)\colon\ j = 1, 2, 3\}$.

**Theorem 1.** *The Comparability Augmentation Problem on the class* $\mathscr{W}$ *is* NP-*complete.*

**Proof.** Consider the following instance $W$ of the Comparability Augmentation Problem associated to an instance of the Partition Problem. Let for $j = 1, 2, 3$ each $B^j$ consist of a single vertex $b_1^j$ whose weight is $s$, and let each $A^j$ include $n$ vertices $a_1^j, \ldots, a_n^j$ whose weights are $w_1, \ldots, w_n$. We are going to show that the instance of the Partition Problem has 'yes' answer if and only if the Comparability Augmentation Problem has a solution with maximum clique weight $3s$.

Let us first assume that the Partition Problem has 'yes' answer. Let $w_1, \ldots, w_r$ be the elements of $A_1$ and $w_{r+1}, \ldots, w_n$ the elements of $A_2$. In the graph $W$ we add the following set of edges: $\{(a_i^2, b_1^1)\colon\ 1 \leqslant i \leqslant r\} \cup \{(a_i^2, b_1^3)\colon\ r+1 \leqslant i \leqslant n\}$. The augmented graph is a comparability graph whose maximum weighted clique has weight $3s$. (To obtain a transitive orientation, we can take $b_1^1 \to b_1^2 \to b_1^3, b_1^1 \to a_i^1, a_i^3 \to b_1^3$ for all $1 \leqslant i \leqslant n$, and $a_j^2 \to b_1^2 \to a_i^2$ for $1 \leqslant i \leqslant r < j \leqslant n$.)

Conversely, let us assume that the Comparability Augmentation Problem has a solution with weight $3s$ on $W$. This means that a set of edges has been added to $W$ in such a way that the maximum weight of the cliques has not been increased in the augmented graph $W^+$. Consider a transitive orientation of $W^+$. We may assume, without loss of generality, that the orientation on $\{b_1^1, b_1^2, b_1^3\}$ is $b_1^1 \to b_1^2 \to b_1^3$. Denote $A^- = \{a_j^2 \in A^2 : b_1^2 \to a_j^2\}$ and $A^+ = \{a_j^2 \in A^2 : a_j^2 \to b_1^2\}$. Observe that, by transitivity, both sets $A^- \cup \{b_1^1, b_1^2\}$ and $A^+ \cup \{b_1^2, b_1^3\}$ are complete subgraphs, with total weight sum $6s$. Thus, the only possibility to not increase the weight $3s$ of maximum cliques is that both $A^-$ and $A^+$ have total weight equal to $s$. This implies a 'yes' answer to the Partition Problem. □

**Corollary 1.** *The Comparability Augmentation Problem is* NP-*complete on stars of cliques.*

**Theorem 2.** *The Comparability Augmentation Problem is* NP-*complete on the class* $\{S^n : n \geqslant 3\}$.

**Proof.** For each instance of the Partition Problem, we define a weight function $w$ on $S^n$ as

$$w(a_i) = w_i \quad \text{for } 1 \leqslant i \leqslant n,$$

$$w(b_i) = s \quad \text{for } 1 \leqslant i \leqslant n.$$

We first show that if there exists a solution for the Partition Problem, then $S^n$ has an augmentation with maximum clique weight $2s$. Let us assume $A_1 = \{w_1, \ldots, w_r\}$ and $A_2 = \{w_{r+1}, \ldots, w_n\}$. Consider the following orientation on $S^n$:

$$a_j \to a_k \quad \text{for } 1 \leqslant j < k \leqslant n,$$

$$a_j \to b_j \quad \text{for } 1 \leqslant j \leqslant r,$$

$$b_j \to a_j \quad \text{for } r < j \leqslant n.$$

In this orientation, the heaviest directed path ending in $b_j$ $(j \leqslant r)$ is $a_1 \to \cdots \to a_j \to b_j$ and it has weight at most $w_1 + \ldots + w_r + s = 2s$. Similarly, the heaviest directed path starting at $b_j$ $(j > r)$ is $b_j \to a_j \to a_{j+1} \to \cdots \to a_n$ and it has weight at most $s + w_{r+1} + \cdots + w_n = 2s$. Finally, the heaviest path not containing any $b_j$ is $a_1 \to a_2 \to \cdots \to a_n$, having weight 2s.

Conversely, now we show that if the Partition Problem has 'no' answer, then any solution of the Comparability Augmentation Problem has weight $> 2s$. Consider an orientation on $S^n$, which defines a transitive orientation on the clique $\{a_1, \ldots, a_n\}$ with a directed Hamiltonian path, say $a_1 \to a_2 \to \cdots \to a_n$. Since there is no solution of the Partition Problem, there is a subscript $i$ with the following properties:

$$w_1 + \cdots + w_i < s, \tag{1}$$

$$w_1 + \cdots + w_{i+1} > s. \tag{2}$$

Consider the edge $(a_{i+1}, b_{i+1})$. If its orientation is $a_{i+1} \to b_{i+1}$, then $S^n$ contains the path $a_1 \to \cdots \to a_{i+1} \to b_{i+1}$ and this path has weight $> 2s$ by (2). On the other hand, if we have $b_{i+1} \to a_{i+1}$, then the path $b_{i+1} \to a_{i+1} \to a_{i+2} \to \cdots \to a_n$ has weight $s + (2s - (a_1 + \cdots + a_i)) > 2s$ by (1).

Thus, in conclusion, $S^n$ has an augmentation of weight $\leqslant 2s$, i.e. of weight precisely $2s$, if and only if $\{w_1, \ldots, w_n\}$ can be partitioned into two subsets with equal sum. □

**Corollary 2.** *The Comparability Augmentation Problem is* NP-*complete on split graphs.*

**Proof.** This follows from the fact that each graph $S^n$ is a split graph. □

## 4. Approximations for split graphs

Due to the complexity results, it is of great interest to derive easily computable upper bounds on the optimum value of the objective function.

**Theorem 3.** *If $\mathscr{G}$ is a split graph, then $M^*(\mathscr{G}) < 2C_{\max}(\mathscr{G})$. Moreover, for every $\varepsilon > 0$, there is a split graph $\mathscr{F}$ such that $M^*(\mathscr{F}) > (2-\varepsilon)C_{max}(\mathscr{F})$.*

**Proof.** Let $\mathscr{G} = (V, E)$, $V = K \cup S$, where $K$ induces a clique and $S \neq \emptyset$ is independent. (For $S = \emptyset$, $\mathscr{G}$ is a complete graph plus possibly some isolated vertices, and $M^*(\mathscr{G}) = C_{\max}(\mathscr{G})$ holds.) Define $\mathscr{H} = (V, H)$ as the graph with edge set $\binom{V}{2} \setminus \binom{S}{2}$, where the notation $\binom{X}{2}$ stands for the collection of all 2-element subsets of a set $X$. Taking a transitive orientation on $\binom{K}{2}$ and orienting all edges from $S$ to $K$, the weighting $p : V \to N$ augments a path to have total weight at most

$$\sum_{v \in K} p(v) + \max_{w \in S} p(w).$$

Take $x \in S$ such that $p(x) = \max_{w \in S} p(w)$ and let $y \in K$ be an arbitrary vertex adjacent to $x$. Then

$$C_{\max}(\mathscr{G}) \geqslant \max\{p(x) + p(y), \sum_{v \in K} p(v)\}$$

and, therefore,

$$M^*(\mathscr{G}) \leqslant p(x) + \sum_{v \in K} p(v) \leqslant 2C_{\max} - p(y) < 2C_{\max}.$$

In order to prove that $M^*(\mathscr{G}) \leqslant (2-\varepsilon)C_{\max}(\mathscr{G})$ does not hold for any $\varepsilon > 0$ in general, consider the following split graph $\mathscr{F}_n = (V^n, F^n)$:

$$V^n := K^n \cup S^n, \quad K^n = \{x_i\colon\ 1 \leqslant i \leqslant n\}, \quad S^n = \{y_{i,j}\colon\ 1 \leqslant i < j \leqslant n\},$$

$$F^n = \binom{K^n}{2} \cup \{(y_{i,j}, x_i),\ (y_{i,j}, x_j)\colon\ 1 \leqslant i < j \leqslant n\}.$$

Moreover, let $q^n : V^n \to N$ be the weight function with $q^n(x_i) = 1$, for $1 \leqslant i \leqslant n$ and $q^n(y_{i,j}) = n$ for $1 \leqslant i < j \leqslant n$. For $n \geqslant 3$, the cliques of $V^n$ are $K^n$ and the sets $\{y_{i,j}, x_i, x_j\}$ and, therefore, obviously, $C_{\max}(V^n) = n + 2$.

Let now $\mathscr{H}$ be a comparability supergraph of $\mathscr{F}^n$ with an optimal transitive orientation on its edge set, i.e. with total weight $M^*$ on the heaviest directed path. We are going to find a lower bound on $M^*$.

Assume, without loss of generality, that $x_1 \to x_2 \to \cdots \to x_{n-1} \to x_n$ is the (unique) directed Hamiltonian path in the subgraph induced by $K$. Since the orientation is acyclic, each $y_{i,i+1}$ admits one of the following three possibilities:

(a) $x_i \to y_{i,i+1}$ and $y_{i,i+1} \to x_{i+1}$;

(b) $y_{i,i+1} \to x_i$ and $y_{i,i+1} \to x_{i+1}$;

(c) $x_i \to y_{i,i+1}$ and $x_{i+1} \to y_{i,i+1}$.

Observe that if case (a) applies to any one particular $i$ $(1 \leqslant i < n)$, then

$$x_1 \to \cdots \to x_i \to y_{i,i+1} \to x_{i+1} \to \cdots \to x_n$$

is a directed path of total weight $2n$. Hence, in such a situation we have $M^* \geqslant 2C_{\max} - 4$, that is

$$\frac{M^*}{C_{\max}} \geqslant 2 - \frac{4}{n+2} > 2 - \varepsilon$$

for $n > 4/\varepsilon - 2$.

Consequently, from now on we may assume that (b) or (c) holds for each $i$, $1 \leqslant i < n$. Define $s(i) = +1$ if case (b) applies and let $s(i) = -1$ otherwise. Since the value $s(1) = +1$ — as well as $s(n-1) = -1$ — would yield a directed path of total weight $2n$, we may assume that $s(1) = -1$ and $s(n-1) = +1$.

Since the sign of $s(i)$ changes at least once from minus to plus along the path, we can select $i$ such that $s(i-1) = -1$ and $s(i) = +1$. In this case, however, $y_{i,i+1} \to x_i \to y_{i-1,i}$ is a directed path of total weight $2n + 1$. $\square$

## 5. Approximations for stars of cliques

If $\mathscr{G}$ is a clique star with $m + 1$ cliques $C_0, C_1, \ldots, C_m$, let us call it an $m$-star for short.

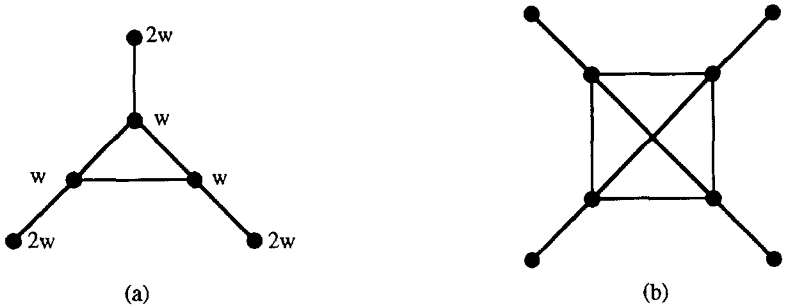



Fig. 1.

**Theorem 4.** *If $\mathscr{G}$ is a* 3*-star or a* 4*-star, then $M^*(\mathscr{G}) \leqslant \frac{4}{3}C_{\max}(\mathscr{G})$, and this upper bound is tight on the class of* 3*-stars.*

**Proof.** Tightness for 3-stars can easily be verified by the graph $\mathscr{G}_0$ shown in Fig. 1(a), with weights $w$ on its triangle and $2w$ on its vertices of degree 1. Indeed, the graph $\mathscr{G}_0$ is a 3-star with $C_{\max} = 3w$. Moreover, in order to augment $\mathscr{G}_0$ to a comparability graph, at least one edge must be added connecting a vertex of the internal triangle with an external vertex of degree 1, hence creating a new triangle of weight $4w$, that is $M^*(\mathscr{G}_0) = 4w$.

We are going to show that even a normal schedule exists in a 3-star or a 4-star with makespan $\leqslant \frac{4}{3}C_{\max}(\mathscr{G})$. Clearly, when restricting our investigation to normal schedules, the complete subgraphs $C_0 \cap C_i$ and $C_i \setminus C_0$ can be represented by single vertices $a_i$ and $b_i$, respectively ($i = 1,2,3,4$), where the weight of a vertex is equal to the sum of the weights in the corresponding complete subgraph. Hence, from now on we assume without loss of generality that $\mathscr{G}$ is the graph shown in Fig. 1(b). Observe that, in order to be in this situation if the graph at the beginning is a 3-star, we may add two dummy vertices $a_4$ and $b_4$ with zero weight.

We note further that the assumption that the weights are integers is irrelevant in the present context because multiplying all weights by any positive number does not change the ratio between the maximum clique weigths in $\mathscr{G}$ and in its extensions. Therefore, in order to simplify one of the steps below, we allow that some of the vertices have fractional weights.

Before going to the main part of the proof, we make a further simplification. Instead of considering all possible comparability graphs $\mathscr{G}^+ \supset \mathscr{G}$, we take only those which, apart from a permutation of the subscripts, admit the following transitive orientation: $a_i \to a_j$ ($1 \leqslant i < j \leqslant 4$), $a_1 \to b_1$, $a_1 \to b_2$, $a_2 \to b_2$, $b_3 \to a_3$, $b_3 \to a_4$, $b_4 \to a_4$ (see Fig. 2). In this orientation just two edges are added to $\mathscr{G}$ (namely $(a_1,b_2)$ and $(b_3,a_4)$, dotted in the figure), and the newly created cliques are the triangles $a_1,a_2,b_2$ and $a_4,a_3,b_3$. It will turn out that even among such extensions of $\mathscr{G}$ there exists one with maximum clique weight $\leqslant \frac{4}{3}C_{\max}(\mathscr{G})$. In these restricted extensions of $\mathscr{G}$, denote by $M_1^*(\mathscr{G})$ the smallest possible value of a maximum weighted clique.

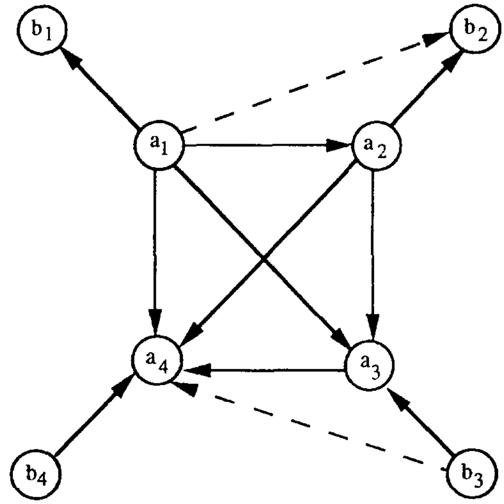



Fig. 2.

It will suffice to show $M_1^*(\mathscr{G}) \leqslant \frac{4}{3}C_{\max}(\mathscr{G})$, since $M_1^*(\mathscr{G}) \geqslant M^*(\mathscr{G})$ clearly holds. We prove the upper bound by contradiction. Suppose that $M_1^*(\mathscr{G}) > \frac{4}{3}C_{\max}(\mathscr{G})$ and let the number of vertices with nonzero weight in $\mathscr{G}$ be as small as possible. Observe that if this number were too small, then we would have $M^*(\mathscr{G}) = C_{\max}(\mathscr{G})$, and the proof would be done. More explicitly,

(1) If both $a_1$ and $a_4$ have zero weight, then neither $(a_1, b_2)$, nor $(b_3, a_4)$ can create a clique weight larger than $C_{\max}(\mathscr{G})$.

(2) The same property holds if both $b_2$ and $b_3$ have zero weight.

(3) If $a_4$ has zero weight, then $M_1^*(\mathscr{G}) > \frac{4}{3}C_{\max}(\mathscr{G})$ remains valid even if we set the weight of $b_4$ to zero.

Based on the above observations, we can assume that either all vertices of $\mathscr{G}$ have positive weights, or precisely $a_4$ and $b_4$ have zero weights, or $b_4$ is the unique zero-weight vertex.

Let $\mathscr{G}^+$ be the extension of $\mathscr{G}$ corresponding to the permutation of the subscripts such that $C_{\max}(\mathscr{G}^+)$ is minimum. Since $\mathscr{G}^+$ is of a restricted type (but transitively oriented), we have $C_{\max}(\mathscr{G}^+) = M_1^*(\mathscr{G}) > \frac{4}{3}C_{\max}(\mathscr{G})$. Denote by $\mathscr{G} - x$ and $\mathscr{G}^+ - x$ the graphs obtained from $\mathscr{G}$ and $\mathscr{G}^+$, respectively, maintaining the same vertices and edges, but in which each positive weight $w$ at $a_i$ is replaced by $w - x$ and each positive weight $w'$ at $b_i$ is replaced by $w' - 2x$.

Observe that during this modification, the weight of each maximal clique of $\mathscr{G}$, as well as of $\mathscr{G}^+$, is decreased by at least $3x$ and by at most $4x$. Thus,

$$M_1^*(\mathscr{G} - x) = C_{\max}(\mathscr{G}^+ - x) \geqslant C_{\max}(\mathscr{G}^+) - 4x$$

and

$$C_{\max}(\mathscr{G} - x) \leqslant C_{\max}(\mathscr{G}) - 3x.$$

Consequently, the assumption $M_1^*(\mathscr{G}) > \frac{4}{3}C_{\max}(\mathscr{G})$ implies that $M_1^*(\mathscr{G}-x) > \frac{4}{3}C_{\max}(\mathscr{G}-x)$ as well, for every $x > 0$. Now, we choose $x$ to be the largest value such that all vertices of $\mathscr{G} - x$ have nonnegative weights. Then there exists a vertex whose weight is positive in $\mathscr{G}$ but zero in $\mathscr{G} - x$. Hence, by the minimality of nonzero weights in $\mathscr{G}$, we should have $M_1^*(\mathscr{G} - x) \leqslant \frac{4}{3}C_{\max}(\mathscr{G} - x)$. This contradiction completes the proof. □

For arbitrary stars of cliques the upper bound of Theorem 4 does not remain valid, as shown by the following observation.

**Proposition 1.** *For every $\varepsilon > 0$ there is a star of cliques, $\mathscr{G}$, with $M^*(\mathscr{G}) > (\frac{3}{2} - \varepsilon)C_{\max}(\mathscr{G})$. More specifically, for every $m \geqslant 3$ there exists an m-star $\mathscr{G}$ such that $M^*(\mathscr{G}) = (1 + \lceil m/2 - 1\rceil/m)C_{max}(\mathscr{G})$.*

**Proof.** Consider the graphs $S^m$ with vertex set $\{a_1, b_1, a_2, b_2, \ldots, a_m, b_m\}$ and edge set $\{(a_i, a_j)\colon 1 \leqslant i < j \leqslant m\} \cup \{(a_i, b_i)\colon 1 \leqslant i \leqslant m\}$. Hence, $C_0 = \{a_1, \ldots, a_m\}$ and $C_i = \{(a_i, b_i)\}$, for $1 \leqslant i \leqslant m$. Define the weight function $w$ on $V(S^m)$ as $w(a_i) = 1$ and $w(b_i) = m - 1$ for $1 \leqslant i \leqslant m$. Clearly, $C_{\max}(S^m) = m$ (see in Fig. 3(a) the graph for $m = 5$).

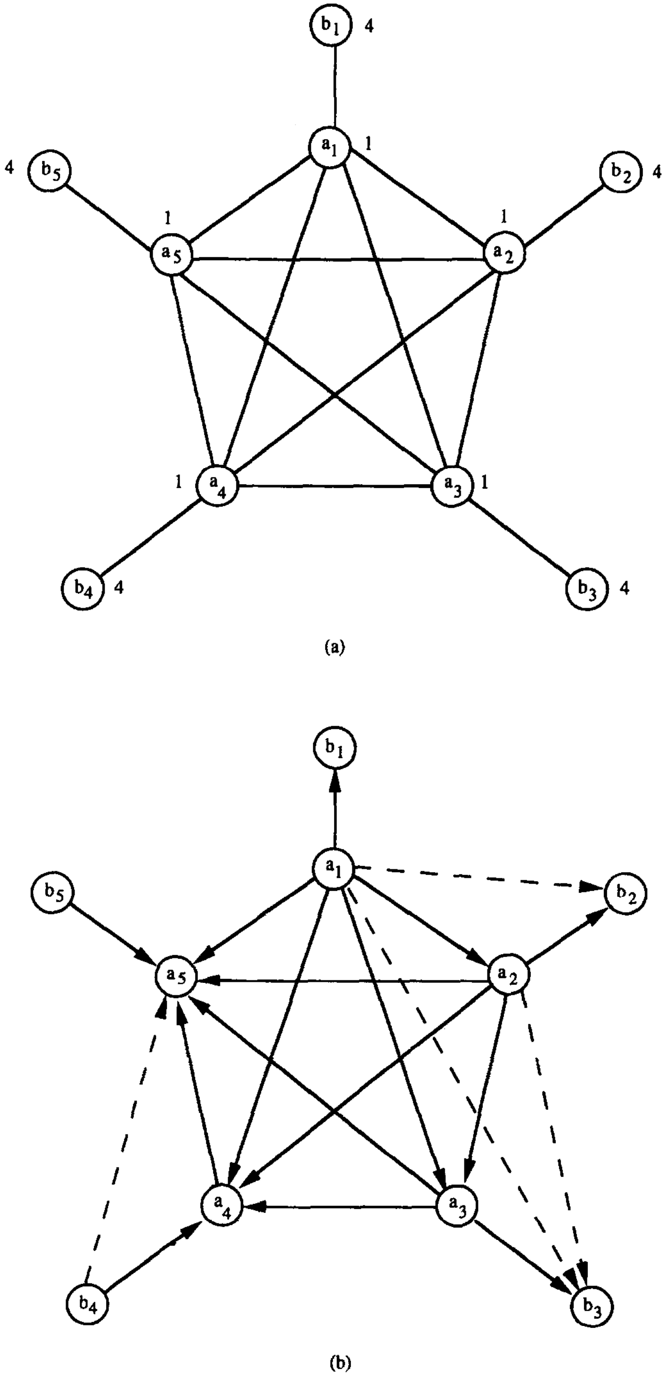
Let $\mathscr{G}^+$ be a transitively oriented extension of $S^m$ with maximum clique weight $M^*(S^m)$. Since the vertices $a_i$ are mutually adjacent, we can assume, without loss of generality, that $a_i \to a_j$ in $\mathscr{G}^+$ if and only if $i < j$. Consider $k = \lceil m/2\rceil$. Then both $a_1, a_2, \ldots, a_k$ and $a_k, a_{k+1}, \ldots, a_m$ are directed paths of at least $\lceil m/2\rceil$ vertices. Thus, no matter how the edge $(a_k, b_k)$ is oriented, the transitive closure of $S^m$ must contain a clique induced by $b_k$ together with $\lceil m/2\rceil$ vertices $a_i$. This clique has total weight $m - 1 + \lceil m/2\rceil$, implying $M^*(S^m) \geqslant (1 + \lceil m/2 - 1\rceil/m)C_{\max}(S^m)$.

The converse inequality is easily seen by the following construction of $\mathscr{G}^+$. First, orient the edges of $S^m$ as $a_i \to a_j$, for $1 \leqslant i < j \leqslant m$, $a_i \to b_i$, for $1 \leqslant i \leqslant \lceil m/2\rceil$ and $b_i \to a_i$, for $\lceil m/2\rceil + 1 \leqslant i \leqslant m$. Then, the transitive closure $\mathscr{G}^+$ of this orientation satisfies the requirement (see Fig. 3(b)). □

As a generalization of Theorem 4, it can be proved that the previous constructions yield the largest possible values of $M^*(\mathscr{G})/C_{\max}(\mathscr{G})$ on stars of cliques.

**Theorem 5.** *If $\mathscr{G}$ is a star of cliques, then $M^*(\mathscr{G}) < \frac{3}{2}C_{\max}(\mathscr{G})$. Moreover, if $\mathscr{G}$ is an m-star and $m \geqslant 3$ is odd, then $M^*(\mathscr{G}) \leqslant (1 + \lceil m/2 - 1\rceil/m)C_{\max}(\mathscr{G})$. If $m \geqslant 4$ is even and all weights in $C_0$ are positive, then $M^*(\mathscr{G}) < (\frac{3}{2} - 1/(2m-2))C_{\max}(\mathscr{G})$.*

**Proof.** The first assertion $M^* < \frac{3}{2}C_{\max}$ follows from the stronger statements on $m$-stars. To prove the latter, the argument goes along the lines of the proof of Theorem 4, with the difference that the central clique $C_0$ now consists of $m$ vertices. We assume that the result is valid for every $m' < m$ (the case $m = 3$ has already been verified).



b1
4
a1
1
b5
4
b2
4
1
1
a5
a2
1
a4
a3
1
b4
4
b3
4
(a)
b1
a1
b5
b2
a5
a2
a4
a3
b4
b3
(b)


Fig. 3.

First, we consider the case of $m$ odd. We again restrict ourselves to a particular class of transitive extensions $\mathscr{G}^+$. Namely, orient the edges of $\mathscr{G}$ as $a_i \to a_j$, for $1 \leqslant i < j \leqslant m$, $a_i \to b_i$ for $1 \leqslant i \leqslant \lceil m/2 \rceil$, and $b_i \to a_i$ for $\lceil m/2 \rceil + 1 \leqslant i \leqslant m$. Among such extensions, denote by $M_1^*(\mathscr{G})$ the smallest value of a maximum weighted clique. Since $M_1^*(\mathscr{G}) \geqslant M^*(\mathscr{G})$, the proof will be done if we show that $M_1^*(\mathscr{G}) \leqslant (1 + \lceil m/2 - 1 \rceil / m) C_{\max}(\mathscr{G})$.

Supposing on the contrary, we consider a graph $\mathscr{G}$ with the smallest number of nonzero weights such that $M_1^*(\mathscr{G}) > (1 + \lceil m/2 - 1 \rceil / m) C_{\max}(\mathscr{G})$. If $a_i$ has weight zero then we can assume without loss of generality that $b_i$ also has. In this case, we can simply delete $\{a_i, b_i\}$ from $\mathscr{G}$ because the coefficient of $C_{\max}$ — which is $1 + \lceil m/2 - 1 \rceil / m$ for $m$ odd and $\frac{3}{2} - 1/(2m-2)$ for $m$ even — is a nondecreasing function of $m$. (For $m-1$ even, the coefficient in question is the same as for $m-2$ odd.) Hence, we can assume all $a_i$ are positive.

Take an extension $\mathscr{G}^+$ of $\mathscr{G}$ with $C_{\max}(\mathscr{G}^+) = M_1^*(\mathscr{G})$. Now we define the graphs $\mathscr{G} - x$ and $\mathscr{G}^+ - x$ which are obtained from $\mathscr{G}$ and $\mathscr{G}^+$, respectively, mantaining the same vertices and edges and decreasing each positive weight of $a_i$ by $x$ and each positive weight of $b_i$ by $(m-1)x$. Then, $C_{\max}(\mathscr{G} - x) \leqslant C_{\max}(\mathscr{G}) - mx$, while $C_{\max}(\mathscr{G}^+ - x) \geqslant C_{\max}(\mathscr{G}^+) - (m - 1 + \lceil m/2 \rceil)x$ by the structure of $\mathscr{G}^+$. Thus, the inequality $M_1^*(\mathscr{G} - x) > (1 + \lceil m/2 - 1 \rceil) C_{\max}(\mathscr{G} - x)$ remains valid. However, an appropriate choice of $x$ (the largest value not creating negative weights) decreases the number of vertices with nonzero weights. This contradiction completes the proof for $m$ odd.

For $m$ even, the reduction step, i.e. the definition of $\mathscr{G} - x$ and $\mathscr{G}^+ - x$, is the same as above. The only difference is that now we can obtain a strict inequality, because $M_1^*(\mathscr{G} - x) = C_{\max}(\mathscr{G}^+ - x) \geqslant C_{\max}(\mathscr{G}^+) - (m - 1 + m/2)x$ and $C_{\max}(\mathscr{G} - x) \leqslant C_{\max}(\mathscr{G}) - mx$. Thus, the contradiction

$$\frac{M^*(\mathscr{G} - x)}{C_{\max}(\mathscr{G} - x)} \geqslant \frac{C_{\max}(\mathscr{G}^+ - x)}{C_{\max}(\mathscr{G} - x)} > \frac{M^*(\mathscr{G})}{C_{\max}(\mathscr{G})} \geqslant \frac{3}{2} - \frac{1}{2m - 2}$$

can be deduced. Indeed, it suffices to prove

$$\frac{C_{\max}(\mathscr{G}^+) - (\frac{3}{2}m - 1)x}{C_{\max}(\mathscr{G}) - mx} > \frac{C_{\max}(\mathscr{G}^+)}{C_{\max}(\mathscr{G})}$$

for some small $x > 0$. Rearrangement yields that this inequality is equivalent to $C_{\max}(\mathscr{G}^+) > \frac{3}{2} - 1/m C_{\max}(\mathscr{G})$. The latter is always valid, because we have assumed $C_{\max}(\mathscr{G}^+) \geqslant (\frac{3}{2} - 1/(2m-2)) C_{\max}(\mathscr{G})$, and $\frac{3}{2} - 1/(2m-2) > \frac{3}{2} - 1/m$ for all $m > 2$. Consequently, strict inequality follows for $m$ even. □

## 6. Conclusions

In this paper we studied the problem of augmenting a graph to a graph which admits a transitive orientation in such a way that the size of its maximum clique is as small as possible. The problem is equivalent to the interval coloring problem and to a multiprocessor scheduling problem. It is well-known that on comparability graphs the problem can be solved in polynomial time. We showed that on some small subclasses of perfect graphs, namely the split graphs and the stars of cliques (two particular classes of triangulated graphs), the problem remains NP-hard. As on these classes of graphs the calculation of the maximum weighted clique can be done in polynomial time, it is of great interest to determine bounds on how much this value increases when augmenting the graph to a comparability graph. It has been shown that on split graphs the cliques can increase at most by a multiplicative constant 2, while in the case of stars of cliques they can increase at most by 1.5.

## Acknowledgements

Research of the second author was partially supported by the Italian Research Council under Grant 03466CT11/115.23225 and research of the third author was partially supported by the "OTKA" Research Fund of the Hungarian Academy of Sciences, Grants no. 2569 and T-014549.

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